Synchronizing chaotic dynamics with uncertainties based on a sliding mode control design

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The synchronization of two chaotic systems with uncertainties is studied in this paper. A feedback controller is provided based on a sliding mode control design. A kind of extended state observer is used to compensate for the systems' uncertainties, such as the structure difference or parameter mismatching, using only the available synchronizing error. Then the feedback controller becomes physically realizable based on the states of the observer, and can be used to synchronize two continuous chaotic systems. Illustrative examples of the synchronization of Duffing and Van der Pol oscillators as well as two Lorenz systems with parameter mismatching are proposed to show the effectiveness of this method.

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I. INTRODUCTION

In the last decade, chaos synchronization has become a popular research topic arousing interests of physical scientists and electrical engineers [[1,2], and references therein]. Such synchronization strategies have potential applications in several areas such as secure communication [[2,3], and references therein], biological oscillators [4] and animal gaits [5]. It has been shown that two identical chaotic oscillators can be synchronized [1]. However, most of the dynamical systems have model (or parametric) uncertainties. To avoid this problem, some strategies have been recently reported [[3,6], and references therein]. In particular, several authors have reported adaptively estimation techniques [6]. These techniques present an acceptable performance and allow synchronization, although the parameters are not known or they are time varying [7]. But the only drawback of these strategies is that the structure of parameters for a given model must be known. This requirement often leads to very complex feedback schemes [3,6,7]. Although the structure of the parameters can be known in some cases, it would be desirable to have a scheme to achieve synchronization even if the slave oscillator has little prior knowledge about the master systems. Moreover, in many real systems, the synchronization is carried out even though the oscillators are different. For example, biological oscillators are often synchronous even when the master and slave systems are quite different [[7] and references therein]. Consequently, the synchronization of systems with uncertainties, such as different model or parameter mismatching, may play an important role in many fields, including chaotic secure communication.

The aim of this paper is to study the synchronization of chaotic systems with uncertainties based on a sliding mode control design. Sliding mode control is a nonlinear control strategy requiring (1) a switching manifold that prescribes the desired dynamics and (2) a control law such that the system trajectory first reaches the manifold and then stays on it forever [8,9]. Yua, Chen, and Chen proposed a method to control chaotic system based on the sliding mode control design [9]. Liao used variable structure control design to control and synchronize a kind of discrete-time chaotic systems [10]. But system uncertainties were never discussed and the controller in Ref. [10] is not realizable because it needs so much structure information of the transmitter and receiver. In this paper, we first get a feedback controller based on sliding mode control design. To make the controller physically realizable, an extended state observer (ESO [7,11,12]) is used to estimate system information, such as unmeasurable states or model differences, so that the complex control can be translated into reality with less system information.

This paper is organized as follows. In Sec. II, the synchronization problem is stated simply. In Sec. III, the sliding mode control design is used to synchronize two chaotic systems with uncertainties. We present some simulation results in Sec. IV. Finally, we give some concluding remarks in Sec. V.

II. SYNCHRONIZATION OF CHAOTIC SYSTEMS WITH UNCERTAINTIES

Let the chaotic master system be given by the equation [7]

$$\dot{X}=F(X,p), \quad y_M=C_MX, \tag{1}$$

where $X\in R''$ is state vector of the master system, $p\in R'''$ is a parameter vector, and the function $F:R''\times R'''\rightarrow R''$ is a smooth vector field. $y_M\in R$ is the output system (measured state). C_M is a vector of proper length that defines the output channel.

Let us now take a chaotic dynamical system of the same order as that of Eq. (1),

$$Y=G(Y,p')+Bu, \quad y_s=C,Y, \tag{2}$$

where $\mathbf{Y}\in R''$ denotes the state vector of the slave system, $\mathbf{p}'\in R'''$ is a parameter vector, $\mathbf{B}$ is a vector of suitable size that defines the control channel, and $u\in R$ is the control command. The vector $\mathbf{C}$, defines the measurable state of the slave system. Without loss of generality, we can assume that the measurable state is given by $y_s=y_1$, that is, only x_1 and y_1 are available in receiver. So we can assume that $C_M=C_s=(1,0,...,0)$. This is realistic because in most cases only one state is available for feedback from the coding (master) as well as decoding (slave) circuit.

From the control theory viewpoint, the synchronization problem can be seen as follows [7]: define E as E

$=(e_1,...,e_n)^r=(y_1-x_1,...,y_n-x_n)^r$. Then the following system describes the dynamics of the synchronization error:

$$\dot{E}=G(Y,p')-F(X,p)+Bu, \quad y=e_1. \tag{3}$$

In this way, the synchronization problem can be seen as the stabilization of Eq. (3) at the origin. In other words, the problem is to find a feedback control law $u(t)$ such that $\lim E\to 0$ (which implies that $Y\to X$) as $t\to\infty$. It has been shown that it is easy to get a kind of $u(t)$ to guarantee the synchronization when $F=G$, $p=p'$ [1]. Our goal in this paper is to design a proper $u(t)$ to achieve the synchronization without this condition.

First, let us define the invertible change of coordinates $\dot{Z}=\Phi(E)$ such that the error system (3) can be written in the following canonical form,

$$\dot{z}_i=z_{j+1}, \quad 1\leqslant i\leqslant \rho-1, \quad \dot{z}_\rho=A(Z,V)+u,$$

$$\dot{V}_j=\xi(Z,V), \quad \rho\leqslant j\leqslant n, \quad \rho\leqslant n, y=\varphi(Z,V). \tag{4}$$

It has been noted that several systems subjected to chaotic synchronization can be transformed into the canonical form (4) [7,12]. For example, the Lorenz dynamical can be transformed into the canonical form with a relative degree $\rho<n$. And nonautonomous second-order chaotic system such as the Duffing oscillator can be written as the canonical form with $\rho=n$ [7].

The following must be noted: (i) if $\rho=n$, the transformed system (4) is the so-called fully linearizable nonlinear system, and (ii) if $\rho<n$, the system (4) is called a partially linearizable nonlinear system. In addition, if the dynamical subsystem $\dot{V}_1=\xi(0,V)$ is asymptotically stable, we say that the system (4) is minimum phase [7]. The two kinds are as follows.

(1) $\rho=n$ (such as Duffing oscillators). Equation (4) has the following canonical form ($z_i=e_i=y_i-x_i$):

$$\dot{z}_i=z_{i+1}, \quad 1\leqslant i\leqslant n-1,..., \quad \dot{z}_n=A(Z)+u, \quad y=\varphi(Z), \tag{5}$$

where $A(\cdot)$ represents systems' structure information.

(2) $\rho<n$ (such as Chua's and Lorenz systems).

Now the synchronizing error system cannot be transformed into the form as Eq. (5). But several chaotic systems are so-called minimum phase, that is, the zero dynamics $\xi(0,V)$ converges to an attractor. In other words, the closed system is internally stable [7,11,12]. From the control viewpoint this is a strong assumption. But this is reasonable for the boundness of chaotic attractor in state space and the interaction of all the trajectories inside the attractor. So when we have taken actions to achieve $\lim z_i\to 0$, $i=1,2,...,\rho$, the part $\xi(Z,V)\to\xi(0,V)\to 0$ asymptotically for the so-called minimum-phase character (see Appendix A for an illustrative example). This is why in many cases, we only need to synchronize one or part of states and the others will be synchronized automatically. So we only consider the condition of $\rho=n$.

III. THE APPLICATION OF SLIDING MODE CONTROL DESIGN IN CHAOTIC SYNCHRONIZATION

For the synchronizing error system (5), we have two assumptions: (1) only z_1 is measurable and (2) $A(\cdot)$ is uncertain. The first assumption is realistic. For instance, in the secure communication case, only the transmitted signal (x_1) and receiver signal (y_1) are available for feedback from measurements [11]. Concerning the second assumption, we claim that it is a general and practical situation because the term $A(\cdot)$ involves the uncertainties in the master as well as slave system. The sources of such uncertainties could be parameter mismatching, unknown initial conditions, or structural differences between models of master and slave systems [13].

The aim of synchronization under these assumptions is to design a physically realizable controller u to achieve $\lim z_i\to 0$, $i=1,2,...,n$. Using the concept of extended systems, the standardized state-space equation of the error states [Eq. (5)] can be obtained as

$$\dot{z}_i=z_{i+1}, \quad 1\leqslant i\leqslant n-1, \quad \dot{z}_n=z_{n+i}, \quad \dot{z}_{n+1}=\Xi(Z,u)+\dot{u}, \tag{6}$$

where

$$\Xi(\cdot)=\sum_{k=1}^{n-1} z_{k+1}\partial_k A(\cdot)+(A(\cdot)+u)\partial_n A(\cdot), \partial_k A(\cdot)$$
$$=\partial A/\partial z_l, \quad k=1,2,...,n.$$

Here we lump the systems' uncertain information $A(\cdot)$ into a new state z_{n+1}. According to sliding mode control design, the sliding surface is defined as [14]

$$S=z_{n+1}-z_{0(n+1)}+\int_0^t\sum_{j=1}^{n+1} c_j z_i dt=0, \tag{7}$$

where $z_{0(n+1)}$ is the initial state of z_{n+1}. Equation (7) can also be formulated as

$$\dot{z}_{n+1}=-\sum_{j=1}^{n+1} c_j z_j \tag{8}$$

with the initial condition $z_{n+1}(0)=z_{0(n+1)}$. Therefore the sliding mode dynamics (the desired dynamics) can be described as

$$\dot{z}_i=z_{i+1}, \quad 1\leqslant i\leqslant n, \quad \dot{z}_{n+1}=-\sum_{j=1}^{n+1} c_j z_j. \tag{9}$$

Or in a matrix equation form as $\dot{Z}=AZ$, where $Z=(z_1,\ldots,z_{n+1})^T$,

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ -c_1 & -c_1 & \cdots & \cdots & -c_{s+1} \end{bmatrix}, \tag{10}$$

initial states being $Z(0)=[z_1(0),\ldots,z_{n+1}(0)]^T$.

The design of c_j, $j=1,...,n+1$ can be determined by choosing the eigenvalues of A such that the corresponding characteristic polynomial equation: $p(n+1)=s^{n+1}+c_{n+1}s''+\cdots+c_1$ is Hurwitz.

According to sliding mode control design, we use the reaching law introduced by Gao *et al.* as [15]

$$\dot{S}=\alpha S-\beta\,\mathrm{sgn}(S), \tag{11}$$

where $0\leqslant\alpha<1$. $\mathrm{sgn}(\cdot)$ denotes the signum function and the switching gain $\beta>0$ is determined such that the sliding condition is satisfied and sliding mode motion will occur.

From Eq. (7) and Eq. (11), it can be found that

$$\alpha S-\beta\,\mathrm{sgn}(S)=\dot{z}_{n+1}+\sum_{j=1}^{n+1}c_jz_j \tag{12}$$

or, alternatively

$$\dot{z}_{n+1}=\Xi(Z,\eta,u)+\dot{u}=\alpha S-\beta\,\mathrm{sgn}(S)-\sum_{j=1}^{n+1}c_jz_j. \tag{13}$$

So the differential equation of control signal u is

$$\dot{u}=\alpha S-\beta\,\mathrm{sgn}(S)-\sum_{j=1}^{n+1}c_jz_j-\Xi(Z,\eta,u), \tag{14}$$

which results in

$$u(t)=\int_0^t\left[\alpha S-\beta\,\mathrm{sgn}(S)-\sum_{j=1}^{n+1}c_jz_j-\Xi(Z,\eta,u)\right]dt. \tag{15}$$

In general, the initial state of the control u is zero. Here a large β is important for the realization of synchronization, which is associated with the system information of the two chaotic systems. We can qualitatively analyze this question with Lyapunov theory as follows.

Substituting control law (14) into the extended system (6), the closed-loop system dynamical can be described as

$$\dot{z}_i=z_{i+1},\quad 1\leqslant i\leqslant n-1,\quad \dot{z}_n=z_{n+1},$$
$$\dot{z}_{n+1}=\alpha S-\beta\,\mathrm{sgn}(S)-\sum_{j=1}^{n+1}c_jz_j, \tag{16}$$

Define the Lyapunov function as $V=\frac{1}{2}S^2$, then its first derivative with respect to time is

$$\begin{aligned}\dot{V}&=S\left(\dot{z}_{n+1}+\sum_{j=1}^{n+1}c_jz_j\right)\\&=S[\alpha S-\beta\,\mathrm{sgn}(S)]\\&=\alpha S^2-\beta abs(S)\leqslant abs(S)[abs(S)-\beta].\end{aligned} \tag{17}$$

From Eq. (7), we know that $S=L(z_1,\ldots,z_{n+1})=M(e_1,\ldots,e_n)=N(X,Y)$. For the boundness of chaotic attractor, we know that S is bounded. So a large enough β will lead to $\dot{V}\leqslant 0$. Here we get the same conclusion as that in [9]. In many situations, condition $\dot{V}\leqslant 0$ can be satisfied by choosing a large enough switching gain β. If system uncertainties can be estimated more accurately, then the resulting control input will be more accurate [8].

From Eqs. (7) and (15), we know the control law is not physically realizable because it requires the measurement of the state z_1, $i=1,...,n$ (that is, the measurement of X,Y) and the system structure information $\Xi(\cdot)$, which is a very stringent demand for the literature of chaotic secure communication. In order to increase the security of communication, the least possible information about the transmitter should be contained in the communication channel. In this paper, we assume that only x_1 is available in receiver, that is, only z_1 is measurable and $\Xi(\cdot)$, which represents the structure information, is uncertain. So a special way must be used to estimate $\Xi(\cdot)$ and z_i, $i=2,...,n$ based on the available signal (z_1 in this paper) to make the controllers (7) and (15) physically realizable. Based on the extended state observer theory [11], excavating information wrapped in measurable synchronizing error (z_1), we use the following ESO to solve this problem:

$$\begin{aligned}&\dot{\hat{z}}_1=\hat{z}_{i+1}-\theta_1\phi_1(\hat{z}_1-z_1),\quad 1\leqslant i\leqslant n-1,\quad \dot{\hat{z}}_n=\hat{z}_{n+1}\\&\quad-\theta_n\phi_n(\hat{z}_1-z_1),\quad \dot{\hat{z}}_{n+1}=-\theta_{n+1}\phi_{n+1}(\bar{z}_1-z_1).\end{aligned} \tag{18}$$

Defining the estimating error as $\omega_1=\hat{z}_1-z_1$, $\omega_2=\hat{z}_2-z_2,\ldots,\omega_{n+1}=\hat{z}_{n+1}-z_{n+1}$, we get the following system:

$$\begin{aligned}&\omega_i=\omega_{i+1}-\theta_i\phi_i(e_1),\quad 1\leqslant i\leqslant n-1,\quad \dot{\omega}_n=\omega_{n+1}\\&\quad+\theta_n\phi_n(e_1),\quad \dot{\omega}_{n+1}=-\Xi(\cdot)-\theta_{n+1}\phi_{n+1}(e_1)\end{aligned} \tag{19}$$

(assume $u=0$). Appropriately choosing parameter θ_i and function $\phi_i(\cdot)$ $(i=1,...,n+1)$, ω_1 will be stabilized at zero. Then $\hat{Z}=(\hat{z}_1,\ldots,\hat{z}_n)$ and $\hat{z}_{n+1}$ will converge to $z_i, i=1,...,n$ and $\Xi(\cdot)$, respectively. Here we choose the following form [11]:

$$\theta_i=L'\lambda_i,\quad \phi_i(\hat{z}_1-z_1)=[abs(\hat{z}_1-z_1))^p\,\mathrm{sgn}(\hat{z}_1-z_1),$$
$$p>0,\quad i=1,2,...,n, \tag{20}$$

where L is the so-called high-gain parameter, which can be interpreted as the uncertainties estimation rate and often be chosen as a constant [7,11]. In order to determine λ_i, we define the variables,

$$\upsilon_1=L''[abs(\hat{z}_1-z_1)]^p\,\mathrm{sgn}(\hat{z}_1-z_1),$$
$$\upsilon_i=L^{n+1-1}(\hat{z}_i-z_1),\quad 1<i\leqslant n,$$
$$\upsilon_{n+1}=(\hat{z}_{n+1}-z_{n+1}).$$

Notice that $\dot{\upsilon}_1=L''[abs(\hat{z}_1-z_1)\mathrm{sgn}(\hat{z}_1-z_1)](\hat{z}_1-\dot{z}_1)$, where superscript $[\cdot]$ means first derivative of time. From the boundness of chaotic attractor, we know that z_1 is bounded. In order to achieve the tracking of z_1, $\hat{z}_1$ must be bounded too. So we can conclude that $[abs(\hat{z}_1-z_1)\mathrm{sgn}(\hat{z}_1-z_1)]$ is bounded. Without loss of generality, assuming its upper limit is χ, which can be chosen as a large enough number, then we get $\dot{v}_1 \leqslant L''\chi(\dot{\hat{z}}_1-\dot{z}_1)$. So we have the following "estimating error" system:

$$\dot{\bar{v}} = LT(\chi,\lambda_1,\ldots,\lambda_{n+1})\overline{v}+\Omega(\cdot), \tag{21}$$

where $\overline{v}=(v_1,\ldots,v_{n+1})^T$, $\Omega(\cdot)=[0,0,...,\Xi]^T$, and T is as follows:

$$T(\chi,\lambda,\ldots,\lambda_{n+i1})=\begin{bmatrix} -\lambda_1\chi & \chi & 0 & \cdots & 0 \\ -\lambda_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -\lambda_n & 0 & 0 & \cdots & 1 \\ -\lambda_{n+1} & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{22}$$

After choosing χ according to experience, we choose the constants λ_i, $1 \leqslant i \leqslant n+1$ in such a way that $T(\cdot)$ has all its eigenvalues in the left-half complex plane. Since X,Y belong to some chaotic attractor, then $A(\cdot)$ in Eq. (5) is a bounded function. Hence $\Xi(\cdot)$ also is bounded function. After choosing proper λ_i, $1 \leqslant i \leqslant n+1$ so that all eigenvalues of $T(\cdot)$ are located in left-half complex plane, we can conclude that $\lim v_i \to 0$, $1 \leqslant i \leqslant n+1$ [7,11,12]. That is, the "estimating error" system $\overline{v}$ is globally asymptotically stable at zero, which implies that $\hat{z}_1 \to z_1$, $1 \leqslant i \leqslant n+1$. So we can get the information of unmeasurable state from $\hat{z}_i$, $1 \leqslant i \leqslant n$ and model uncertainties from $\hat{z}_{n+1}$. Note that in Eq. (18), $\hat{z}_{n+1}$ represents the system model uncertainties $\Xi(\cdot)$, so z_{n+1} in Eq. (15) is equal to $A(\cdot)+u=\hat{z}_{n+1}+u$. Then Eqs. (7) and (15) become

$$S=\hat{z}_{n+1}+u\sim\hat{z}_{(n+1)}+\int_0^i\left[\sum_{j=1}^{n} c_j\hat{z}_j+c_{n+1}(\dot{z}_{n+1}+u)\right]dt, \tag{23}$$

$$u(t)=\int_0^i\left\{\alpha S-\beta\,\mathrm{sgn}(S)-\left[\sum_{j=1}^{n} c_j\hat{z}_j+c_{n+1}(\hat{z}_{n+1}+u)\right]-\dot{\hat{z}}_{n+1}\right\}dt. \tag{24}$$

Notice that control law (23) and (24) only use the estimation of structure information (by means of $\hat{z}_{n+1}$) and $\dot{Z}$, which are provided by estimator (18). And the dynamical compensator (18) only uses the measurable synchronizing error (z_1 in this paper). So Eqs. (23) and (24) neglect the system uncertainties and are more physically realizable than Eqs. (7) and (15) do. After the complete synchronization of two systems, many ways can be used to transmit information, such as additive signal masking [17], inverse system masking (ISM) [18], and so on.

IV. NUMERICAL STUDIES

A. Synchronization of two strictly different systems

In this section, the synchronization of Duffing and Van der Pol oscillator is presented to show the effectiveness of this design mentioned above.

The two order master Duffing system is described as follows:

$$\dot{x}_1=x_2,$$
$$\dot{x}_2=1.8x_1-0.1x_2-x^3{}_1+1.1\cos(0.4t). \tag{25}$$

The same order slave system Van der pol [7] oscillator is as follows:

$$\dot{y}_1=y_2,$$
$$\dot{y}_2=-0.1(1-y_1)y_2-y_1{}^3+0.3\cos(1.0t)+u, \tag{26}$$

where u is the controller needed to be chosen. From Eqs. (25) and (26), the synchronizing error dynamical (y_1-x_1,y_2-x_2) can be described as the canonical form

$$\dot{e}_1=e_2,\dot{e}_2=\Theta+u, \tag{27}$$

where $\Theta=-0.1(1-y_1)y_2-y_1{}^3+0.3\cos(1.0t)-1.8x_1+0.1x_2+x^3{}_1-1.1\cos(0.4t)$ contains two systems' model difference, which is unknown to us. So the extended state observer [Eq. (18)] can be described in the following form $[\hat{Z}=(\hat{z}_1,\hat{z}_2,\hat{z}_3)^T]$:

$$\dot{\hat{z}}_1=\hat{z}_2-L\lambda_1(abs(\hat{z}_1-z_1))^p\,\mathrm{sgn}(\hat{z}_1-z_1),$$
$$\dot{\hat{z}}_2=\hat{z}_3-L^2\lambda_2(abs(\hat{z}_1-z_1))^p\,\mathrm{sgn}(\hat{z}_1-z_1), \tag{28}$$
$$\dot{\hat{z}}_3=-L^3\lambda_3(abs(\hat{z}_1-z_1))^p\,\mathrm{sgn}(\hat{z}_1-z_1),$$

where $\hat{z}_3$ represents the structure difference Θ. Hence the sliding control law (23) and (24) can be described as

$$S=\hat{z}_3+u-\dot{z}_3(0)+\int_n^t[c_3(\hat{z}_3+u)+c_2\hat{z}_2+c_1\hat{z}_1]dt, \tag{29}$$

$$u(t)=\int_a^t[\alpha S-\beta\,\mathrm{sgn}(S)-[c_3(\hat{z}_3+u)+c_2\hat{z}_2+c_1\hat{z}_1]-\dot{\hat{z}}_3]dt. \tag{30}$$

Here we randomly choose the systems' initial condition (0.3,0.5), (1,0.61) for Duffing and Van der Pol, respectively. The initial condition for $(\hat{z}_1,\hat{z}_2,\hat{z}_3)$ is randomly chosen as (0, 0, 0.5). The initial condition for Eq. (29) is $S(0)=0$ and $u(0)=0$ for Eq. (30). Here we choose $\chi=4$. According to the parameter-choosing scheme mentioned above, we choose parameters as follows: the switching gain $\alpha=0.1$, $\beta=0.4$, the parameter for $c_i, i=1,2,3$ is as $[c_3,c_2,c_1]=[165,65,3]$ then engivalues of A in Eq. (10) are -2.5814, $-0.2093 \pm 7.9924i$, the high gain is $L=10$, the parameter of λ_1, i

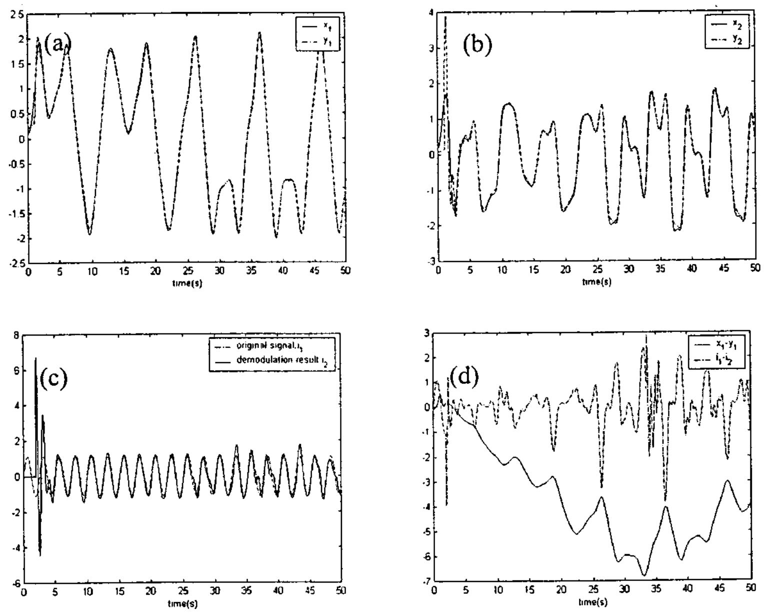

FIG. 1. Simulation results. (a) (x_1,y_1) vs time; (b) (x_2,y_2) vs time; (c) (i_1,i_2) vs time; (d) (x_1-y_1) vs time, (i_1-i_2) vs time when $\beta=0.01$.

$=1,2,3$ is $[\lambda_1,\lambda_2,\lambda_3]=[1,2,3]$ so that engivalues of matrix (22) are -2.6850, $-0.6575\pm 2.0092i$, and $p=0.5$. The control was activated at $t=2$. We use the simulink of MATLAB5.3 to carry out this numeric simulation. Figures 1(a) and 1(b) show the simulation result.

After the synchronization of transmitter and receiver, we can use many ways to transmit information [[17,18], and references therein]. Here we use ISM to discuss it simply. We add information to the right-hand side of x_2. Then the second function of Eq. (25) is

$$\dot{x}_2=1.8_1-0.1x_2-x_1^3+i_1(t), \tag{31}$$

where $i_1(t)$ is the signal to be transmitted. Here we choose $i_1(t)=1.1\cos(0.4t)$ simply. According to ISM [18], we have the demodulator as

$$i_2(t)=\dot{x}_2-1.8x_1+0.1x_2+x_1^3, \tag{32}$$

where $i_2(t)$ is the demodulated signal. After the synchronization of the two systems, we know that $y_2\Rightarrow x_2$ and x_1 is the driving signal that is available at the receiver side. So the demodulator (32) can be described as the realizable form

$$i_2(t)=\dot{y}_2-1.8x_1+0.1y_2+x_1^3. \tag{33}$$

The simulation result is shown in Fig. 1(c), which reveals the good demodulation. From Eq. (17) we know that large β is needed so that the sliding condition is satisfied and sliding mode motion will occur, which guarantees the synchronization and signal demodulation. Figure 1(d) shows that synchronization is not achieved when $\beta=0.01$ and other parameters are the same as above. In the simulation we found that the larger β is, the better the synchronization and signal demodulation. But to find out whether there exists quantitative relation between them needs further work.

B. Synchronization of systems with parameter mismatching

In this section, we study the synchronization of two Lorenz systems with parameter mismatching. For two Lorenz systems $X_{1,2}=(x_{1,2},y_{1,2},m_{1,2})^T$ as follows:

$$\dot{x}_{1,2}=\sigma_{1,2}(y_{1,2}-x_{1,2}), \quad \dot{y}_{1,2}=r_{1,2}x_{1,2}-y_{1,2}-x_{1,2}m_{1,2},$$

$$\dot{m}_{1,2}=x_{1,2}y_{1,2}-\psi_{1,2}m_{1,2}, \tag{34}$$

where subscript 1 means the transmitter and 2 receiver. σ, r, ψ are system parameters. Control u is added on the right-hand side of x_2. So we get the following synchronizing error system:

$$\dot{e}_1=\Delta f_1+u, \quad \dot{e}_2=\Delta f_2, \quad \dot{e}_3=\Delta f_3, \tag{35}$$

where $\Delta F=(\Delta f_1,\Delta f_2,\Delta f_3)$ denotes the uncertainties of these two systems (parameter mismatching). So the canonical form [Eq. (4)] can be described as

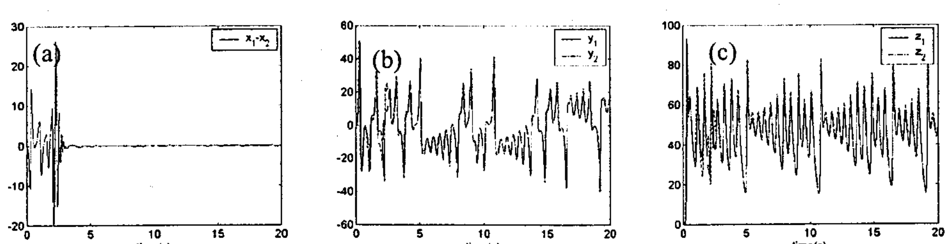

FIG. 2. Simulation results. (a) (x_1-x_2) vs time; (b) (y_1,y_2) vs time; (c) (m_1,m_2) vs time.

$$\dot{z}_\rho = \eta, \quad \dot{\eta} = \Xi(z,\eta,u), \quad \dot{V} = \xi(z,V), \tag{36}$$

where $z_\rho = c_1$, $V = (e_2, e_3)$, $\rho = 1$. It is easy to see that subsystem $\xi(0,V)$ is the so-called minimum-phase. Let us assume that x_1 is the driving signal, so z_1 is measurable. Then we get the extended state observer [Eq. (18)] as the following form $[\hat{Z} = (\hat{z}_1, \hat{z}_2)^T]$:

$$\dot{\hat{z}}_1 = \hat{z}_2 - L\lambda_1[abs(\hat{z}_1 - z_1)]^p \operatorname{sgn}(\hat{z}_1 - z_1),$$

$$\dot{\hat{z}}_2 = -L^2\lambda_2[abs(\dot{z}_1 - z_{11})]^p \operatorname{sgn}(\hat{z}_1 - z_1), \tag{37}$$

where $\hat{z}_2$ represents the uncertainty Δf_1. So the sliding control law [Eqs. (23) and (24)] can be described as

$$S = \hat{z}_2 + u - \hat{z}_2(0) + \int_0^t [c_2(\hat{z}_2 + u) + c_1\hat{z}_1]dt, \tag{38}$$

$$u(t) = \int_0^t [\alpha S - \beta \operatorname{sgn}(S) - \{c_2(\hat{z}_2 + u) + c_1\hat{z}_1\} - \dot{\hat{z}}_2]dt. \tag{39}$$

Here we choose initial condition $S(0) = 0$, $u(0) = 0$. Initial condition for the two Lorenz systems are (0.4,0,0) and (0.5,0.87,1.1). Initial condition for $(\hat{z}_1, \hat{z}_2)$ is randomly chosen as (0,0.5). Let $\chi = 4$, $L = 10$, $p = 0.5$, $[c_1, c_2] = [56,4]$, $[\lambda_1, \lambda_2] = [1,2]$, $\alpha = 0$, $\beta = 0.3$, and $\sigma_1 = 10$, $\beta_1 = 8/3$, $r_1 = 50$, $\sigma_2 = 9$, $\beta_2 = 8.5/3$, $r_2 = 52$. In order to strengthen the effect of control, we use $u' = 10u$ instead of u in this simulation. The control was activated at $t = 2$. Simulation results are shown in Fig. 2.

V. CONCLUSIONS

In this paper, a sliding mode control for synchronizing chaotic systems with uncertainties is proposed. Based on a rigorous mathematical analysis and Lyapunov stability theory, a sliding mode controller is designed such that two chaotic systems with uncertainties can be synchronized. To make this control physically reliable, a kind of extended state observer is used to estimate the systems' model difference and the immeasurable states based on the measurable synchronizing error. Duffing and Van der Pol oscillators and two Lorenz systems were used as examples to verify and visualize this strategy. Simulation results demonstrate that the proposed design is able to achieve the synchronization of two chaotic systems of the same order with little system information. Both analysis and simulations reveal that the proposed sliding mode control design and the ESO have great potential for synchronizing two chaotic systems with uncertainties, which is significant for secure communication. Further works, such as the quantitative relationship between α, β and synchronization performance, the use of this scheme in communication and so on, are going on.

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APPENDIX A: INTERNAL STABILITY OF THE SYNCHRONIZING ERROR SYSTEM OF TWO LORENZ SYSTEMS

Let us consider the two Lorenz systems as follows:

$$\dot{x}_{1,2} = \sigma(y_{1,2} - x_{1,2}), \quad \dot{y}_{1,2} = rx_{1,2} - y_{1,2} - x_{1,2}m_{1,2},$$

$$\dot{m}_{1,2} = x_{1,2}y_{1,2} - \psi m_{1,2}, \tag{A1}$$

where subscript 1 represents the master system, and 2 the slave. The control is added to x_t. The synchronizing error dynamical is as follows ($e_1 = x_1 - x_2$; $e_2 = y_1 - y_2$; $e_1 = m_1 - m_2$):

$$\dot{z}_1 = \Theta(\cdot) + u, \dot{v}_1 = rz_1 - v_1 - z_1v_2, \quad v_2 = z_1v_1 - \psi v_2,$$

$$y = z_1, \tag{A2}$$

where $z_1 = e_1$, $V = (v_1, v_2)^T = (e_2, e_3)^r$, $\rho = 1$. So we get

$$\dot{V} = AV + BS, \tag{A3}$$

where

$$A = \begin{bmatrix} -1 & -z_1 \\ z_1 & -\psi \end{bmatrix}, \quad B = \begin{bmatrix} r \\ 0 \end{bmatrix}, \quad S = z_1.$$

For Lorenz system, the following inequation is satisfied [16]:

$$x^2(t) + y^2(t) + [m(t) - r - \sigma]^2 \leqslant (\sigma + r)^2K^2, \tag{A4}$$

where $K^2 = \frac{1}{4} + (\psi/4)\max(\sigma^{-1},1)$, so z_1 is bounded. For appropriate parameter ψ such that A has all its eigenvalue in the left-half complex plane, then Eq. (A3) is asymptotically stable, that is, $\xi(0,V)$ converges to an attractor. For other chaotic systems, such as Chua's circuit, etc., because of the character of chaotic attractor, we can draw the same conclusion.

[1] L. M. Pecora and T. L. Carriol, Phys. Rev. Lett. **64**, 821 (1990); A. K. Kozlov, V. D. Shalfeev, and L. O. Chua, Int. J. Bifurcation Chaos Appl. Sci. Eng. **6**, 255 (1996); **6**, 569 (1996).

[2] U. Parlitz *et al.*, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 973 (1992); U. Parlitz *et al.*, *ibid.* **4**, 1715 (1994).

[3] C. W. Wu *et al.*, Int. J. Bifurcation Chaos Appl. Sci. Eng. **6**, 455 (1996); M. di Berdnardo, *ibid.* **6**, 557 (1996).

[4] M. Bazhenov, R. Huerta, M. I. Rabinovich, and T. Sejnowski, Physica D **116**, 392 (1998).

[5] J. J. Collins and I. Stewart, J. Nonlinear Sci. **3**, 349 (1995).
[6] L. OlChua *et al.*, Int. J. Bifurcation Chaos Appl. Sci. Eng. **6**, 189 (1996).
[7] Ricardo Femat *et al.*, Physica D **139**, 231 (2000).
[8] H. Sira-Ramirez, Int. J. Control **57**, 1039 (1993).
[9] Her-Terng Yua, Chua'O-Kuang Chen, and Chieh-li Chen, Int. J. Bifurcation Chaos Appl. Sci. Eng. **5**, 1139 (2000).
[10] The-Lu Liao and Nan-Sheng Huang, Phys. Lett. A **234**, 262 (1997).
[11] Han Jingqing, Control and Decision **110(1)**, 85 (1995) (in Chinese).
[12] Ricardo Fernat and Jose Alvarez-Ramirez, Phys. Lett. A **236**, 307 (1997).
[13] V. Toronov and V. Derbov, Phys. Rev. E **55**, 3689 (1997).
[14] C. L. Chen and W. Y. Lin, Proc. Inst. Mech. Eng. **211**, 75 (1997).
[15] W. Gao *et al.*, IEEE Trans. Ind. Electron. **42**, 117 (1995).
[16] H. R. Mettin and A. Scheeling, Phys. Rev. E **51**, 4065 (1995).
[17] K. M. Cuomo and A. V. Oppenheim, Int. J. Bifurcation Chaos Appl. Sci. Eng. **3**, 1629 (1993); K. M. Short, *ibid.* **4**, 959 (1994).
[18] C.W. Lu and L.O. Chua, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4,** 929 (1994); U. Feldmann, M. Hasler, and W. Schwarz, Int. J. Cir. Theor. Appl. **24**, 551 (1996).